

Efficient and Optimal Attitude Determination Using Recursive Global Positioning System Signal Operations

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A new and efficient algorithm is developed for attitude determination from global positioning system signals. The new algorithm is derived from a generalized predictive filter for nonlinear systems. This uses a one time-step ahead approach to propagate a simple kinematics model for attitude determination. The advantages of the new algorithm over previously developed methods include: it provides optimal attitudes even for coplanar baseline configurations, it guarantees convergence even for poor initial conditions, it is a noniterative algorithm, and it is computationally efficient. These advantages clearly make the new algorithm well suited to onboard applications. The performance of the new algorithm is tested on a dynamic hardware simulator. Results indicate that the new algorithm accurately estimates the attitude of a moving vehicle and provides robust attitude estimates even when other methods, such as a linearized least-squares approach, fail due to poor initial starting conditions.

Introduction

PHASE difference measurements from global positioning system (GPS) signals provide a novel approach to determine the attitude of a vehicle. This approach has been successfully applied to air-,¹ sea-,² and space-based^{3,4} vehicles. The problem of finding the attitude of a vehicle using GPS signals essentially involves a two-step process. First, because phase differences are used, the correct number of integer wavelengths between a given pair of antennas must be found. This problem can generally be solved using static integer searches or using motion-based techniques. Much attention has been placed on resolving the integer ambiguity problem over many years, e.g., see Refs. 5–7. Second, once the integer ambiguities are resolved, then the attitude problem must be solved. The solution to this problem poses a difficult task, which has just recently gained attention in the research community.

The most widely used techniques for attitude determination involve methods that solve Wahba's problem.⁸ This problem involves finding a proper orthogonal matrix that minimizes the scalar weighted norm-squared residual between sets of 3×1 body vector observations and 3×1 inertial observations mapped into the body frame. Many methods have been developed that solve this problem accurately and efficiently, e.g., see Refs. 9 and 10. However, the GPS observation is not in the form of a vector observation, so that finding the attitude using GPS signals is inherently more difficult.¹¹

Minimizing the GPS loss function can be accomplished by using nonlinear least-squares or gradient-based search techniques. However, these methods may require a large number of iterations to converge and are not efficient.¹² Cohen's linearized approach¹³ involves finding a small-angle rotation, which maps an initial attitude estimate to the desired attitude matrix. This approach works well for a good initial guess, but is not guaranteed to converge to the correct solution for large initial errors. Other methods^{11,14} convert the GPS loss function into Wahba's form. The transformation has been shown to be exact only when the baselines or sightlines are proportional to an orthonormal basis. Significant errors may arise

if this condition is not true. An extreme example of this scenario is when three baselines are coplanar.

In this paper, a new and efficient algorithm is derived that determines the attitude using GPS observations. The new algorithm is based on a predictive filter scheme for nonlinear systems first introduced by Crassidis and Markley.¹⁵ This scheme uses a recursive (one time-step ahead) method to predict the required model error so that the propagated model produces optimal estimates. The filter developed in this paper is essentially reduced to a deterministic approach because the corrections required to update the model are not weighted in the loss function. The specific name of the new algorithm using GPS signals is attitude-lean-losing-estimator using GPS recursive operations (ALLEGRO). The following lists the main advantages of the ALLEGRO algorithm over previously developed methods:

1) The algorithm is not iterative.

2) It always converges to the correct solution provided that there is a minimum number of baselines and sightlines.

3) The algorithm is easy to implement.

An attitude error covariance expression from the general GPS loss function has been developed by Crassidis and Markley.¹¹ It will be shown that the ALLEGRO algorithm produces estimates that have exactly the same error covariance provided that the observation sampling is fairly frequent. Therefore, the ALLEGRO algorithm minimizes the general GPS loss function.

The organization of this paper proceeds as follows. First, the concept of the GPS phase difference observation is introduced. Then, the general loss function used for GPS-attitude determination is reviewed. Next, for completeness the optimal attitude error covariance derivation is shown. Then, the generalized predictive filter for nonlinear systems is reviewed, followed by an application of this scheme to the GPS loss function. Also, an attitude error covariance expression is derived for the ALLEGRO algorithm. Finally, the algorithm is tested using a GPS hardware simulator.

Background

In this section, a brief background of the GPS phase difference measurement is given. The main measurement used for attitude determination is the phase difference of the GPS signal received from two antennas separated by a baseline. The wave front angle and wavelength are used to develop a phase difference, as shown in Fig. 1. The phase difference is obtained by

$$b_l \cos \theta = \lambda(\Delta\phi - n) \quad (1)$$

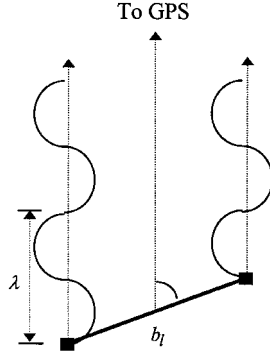
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Fig. 1 GPS wavelength and wave front angle.



where b_l is the baseline length (in centimeters), θ is the angle between the baseline and the line of sight to the GPS spacecraft, n is the integer part of the phase difference between two antennas, $\Delta\phi$ is the fractional phase difference (in cycles), and λ is the wavelength (in centimeters) of the GPS signal. The two GPS frequency carriers are L1 at 1575.42 MHz and L2 at 1227.6 MHz. As of this writing, nonmilitary applications generally use the L1 frequency. Then, assuming that the integer offset is known and compensated, the measured fractional phase difference ($\Delta\tilde{\phi}$) can be expressed by

$$\Delta\tilde{\phi} = \mathbf{b}^T \mathbf{A} \mathbf{s} + v \quad (2)$$

where $\mathbf{s} \in \mathbb{R}^3$ is the normalized line-of-sight vector to the GPS spacecraft in an inertial frame; $\mathbf{b} \in \mathbb{R}^3$ is the baseline vector in wavelengths, which is the relative position vector from one antenna to another; $\mathbf{A} \in \mathbb{R}^{3 \times 3}$ is the attitude matrix, an orthogonal matrix with determinant 1 representing the transformation between the two frames; and v is the measurement error, which is assumed to be a zero-mean stationary Gaussian process with standard deviation given by σ .

Attitude determination using GPS signals involves finding the proper orthogonal matrix \mathbf{A} that minimizes the following generalized loss function:

$$J(\mathbf{A}) = \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} (\Delta\tilde{\phi}_{ij} - \mathbf{b}_i^T \mathbf{A} \mathbf{s}_j)^2 \quad (3)$$

where m is the number of baselines, n now is the number of observed GPS spacecraft, and σ_{ij} is the standard deviation of the ij th measurement error. The standard deviation is $0.5 \text{ cm}/\lambda = 0.026$ wavelengths for typical phase noise.¹³

An attitude error covariance can be derived from the GPS loss function in Eq. (3). This is accomplished by using results from maximum likelihood estimation.^{11,16} The Fisher information matrix for a parameter vector \mathbf{x} is given by

$$F_{xx} = E \left\{ \frac{\partial}{\partial \mathbf{x} \partial \mathbf{x}^T} J(\mathbf{x}) \right\}_{\mathbf{x}_{\text{true}}} \quad (4)$$

where $E\{\cdot\}$ denotes expectation and $J(\mathbf{x})$ is the negative log likelihood function, which is the loss function in this case. If the measurements are Gaussian and linear in the parameter vector, then the error covariance is given by

$$P_{xx} = F_{xx}^{-1} \quad (5)$$

Now, the attitude matrix is approximated by

$$\mathbf{A} = e^{-[\delta\alpha \times]} \mathbf{A}_{\text{true}} \approx (\mathbf{I}_{3 \times 3} - [\delta\alpha \times] + \frac{1}{2} [\delta\alpha \times]^2) \mathbf{A}_{\text{true}} \quad (6)$$

where $\delta\alpha$ represents a small angle error, and $\mathbf{I}_{3 \times 3}$ is a 3×3 identity matrix. The 3×3 matrix $[\delta\alpha \times]$ is referred to as cross-product matrix because $\mathbf{a} \times \mathbf{b} = [\mathbf{a} \times] \mathbf{b}$, with

$$[\mathbf{a} \times] \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (7)$$

Equation (6) is next substituted into Eq. (3) to determine the Fisher information matrix. First-order terms vanish in the partials, and third-order terms are small because we assume the probability distribution to be approximately symmetric about the mean. Also, assuming that the quartic terms are negligible (see Ref. 17 for a Gaussian approximation to fourth-order terms) leads to the following form for the optimal covariance:

$$P_{\text{opt}} \approx \left\{ \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} [\mathbf{A} \mathbf{s}_j \times] \mathbf{b}_i \mathbf{b}_i^T [\mathbf{A} \mathbf{s}_j \times]^T \right\}^{-1} \quad (8)$$

Note that the optimal covariance requires knowledge of the attitude matrix. However, if the baselines are noncoplanar, then the optimal covariance can be determined without the attitude knowledge.¹¹

There are a number of methods available to minimize the GPS loss function shown in Eq. (3), including the standard parameter optimization techniques, such as the gradient method.¹⁸ However, these methods are usually computationally inefficient. A more practical approach uses a linearized least-squares method.¹³ This begins by performing a first-order linearization about a nominal attitude, so that

$$\mathbf{A} = \mathbf{A}_0 (\mathbf{I}_{3 \times 3} + [\delta\theta \times]) \quad (9)$$

where \mathbf{A}_0 represents some nominal attitude, and $\delta\theta$ represents a small angle correction. Then, defining a perturbation equation for the phase difference measurement leads to

$$\delta\phi = \Delta\tilde{\phi} - \Delta\phi_0 = \mathbf{s}^T \mathbf{A}_0^T [\mathbf{b} \times] \delta\theta + v \equiv \mathbf{h}^T \delta\theta + v \quad (10)$$

Equation (10) represents a linearized sensitivity equation between the measured differential carrier phase and the perturbation to the initial attitude guess. All available differential phase measurements can be stacked into a single linearized vector equation, given by

$$\mathbf{H} = \begin{bmatrix} \mathbf{h}_{11}^T \\ \vdots \\ \mathbf{h}_{mn}^T \end{bmatrix} \quad (11a)$$

$$\delta\phi = \mathbf{H} \delta\theta + v \quad (11b)$$

Therefore, Eq. (11b) can be used to find a least-squares estimate of the attitude from the nominal attitude. In practice, the solution is usually obtained iteratively by using the previous epoch as an initial guess. Also, it is easy to see that $(\mathbf{H}^T \mathbf{R}^{-1} \mathbf{H})^{-1}$ is equivalent to the attitude error covariance expression in Eq. (8), where \mathbf{R} is the diagonal covariance matrix of the measurement error process v . The linearized approach provides an efficient method for attitude determination; however, it is sensitive to the initial attitude guess, which may cause divergence problems (as will be shown).

Predictive Attitude Determination

In this section, the ALLEGRO algorithm is derived using a nonlinear predictive approach. First, a brief review of the nonlinear predictive filter is shown (see Ref. 15 for more details). Then, the filter algorithm is reduced to a deterministic-type approach for attitude determination. Finally, a covariance expression for the attitude errors using the ALLEGRO algorithm is derived.

Predictive Filtering

In the nonlinear predictive filter it is assumed that the state and output estimates are given by a preliminary model and a to-be-determined model error vector, given by

$$\hat{\mathbf{x}}(t) = \mathbf{f}[\hat{\mathbf{x}}(t), t] + \mathbf{G}(t) \mathbf{d}(t) \quad (12a)$$

$$\hat{\mathbf{y}}(t) = \mathbf{c}[\hat{\mathbf{x}}(t), t] \quad (12b)$$

where $\mathbf{f} \in \mathbb{R}^p$ is the model vector, $\hat{\mathbf{x}}(t) \in \mathbb{R}^p$ is the state estimate vector, $\mathbf{d}(t) \in \mathbb{R}^l$ is the model error vector, $\mathbf{G}(t) \in \mathbb{R}^{p \times l}$ is the model-error distribution matrix, $\mathbf{c} \in \mathbb{R}^m$ is the measurement vector, and $\hat{\mathbf{y}}(t) \in \mathbb{R}^m$ is the estimated output vector. State-observable

discrete measurements are assumed for Eq. (12b) in the following form:

$$\tilde{\mathbf{y}}(t_k) = \mathbf{c}[\mathbf{x}(t_k), t_k] + \mathbf{v}(t_k) \quad (13)$$

where $\tilde{\mathbf{y}}(t_k) \in \mathbb{R}^m$ is the measurement vector at time t_k , $\mathbf{x}(t_k) \in \mathbb{R}^p$ is the true state vector, and $\mathbf{v}(t_k) \in \mathbb{R}^m$ is the measurement noise vector, which is assumed to be a zero-mean, stationary, Gaussian white-noise distributed process with

$$E\{\mathbf{v}(t_k)\} = \mathbf{0} \quad (14a)$$

$$E\{\mathbf{v}(t_k)\mathbf{v}^T(t_{k'})\} = R\delta_{kk'} \quad (14b)$$

where $R \in \mathbb{R}^{m \times m}$ is a positive-definite covariance matrix.

A loss function consisting of the weighted sum square of the measurement-minus-estimate residuals plus the weighted sum square of the model correction term is minimized, given by

$$J = \frac{1}{2}[\tilde{\mathbf{y}}(t_{k+1}) - \hat{\mathbf{y}}(t_{k+1})]^T R^{-1}[\tilde{\mathbf{y}}(t_{k+1}) - \hat{\mathbf{y}}(t_{k+1})] + \frac{1}{2}\mathbf{d}^T(t_k)W\mathbf{d}(t_k) \quad (15)$$

where $W \in \mathbb{R}^{l \times l}$ is weighting matrix. The necessary conditions for the minimization of Eq. (15) lead to the following model error solution:

$$\mathbf{d}(t_k) = -\left\{[\Lambda(\Delta t)S(\hat{\mathbf{x}}_k)]^T R^{-1}\Lambda(\Delta t)S(\hat{\mathbf{x}}_k) + W\right\}^{-1} \times [\Lambda(\Delta t)S(\hat{\mathbf{x}}_k)]^T R^{-1}[\mathbf{z}(\hat{\mathbf{x}}_k, \Delta t) - \tilde{\mathbf{y}}(t_{k+1}) + \hat{\mathbf{y}}(t_k)] \quad (16)$$

where $\hat{\mathbf{x}}_k \equiv \hat{\mathbf{x}}(t_k)$, Δt is the measurement sampling interval, $S(\hat{\mathbf{x}}) \in \mathbb{R}^{m \times l}$ is a generalized sensitivity matrix, and $\Lambda(\Delta t) \in \mathbb{R}^{m \times m}$ is a diagonal matrix with elements given by

$$\lambda_{ii} = \Delta t^{p_i} / p_i!, \quad i = 1, 2, \dots, m \quad (17)$$

where p_i , $i = 1, 2, \dots, m$, is the lowest order of the derivative of $c_i[\hat{\mathbf{x}}(t)]$ in which any component of $\mathbf{d}(t)$, first appears due to successive differentiation and substitution for $\hat{\mathbf{x}}_i(t)$ on the right side. The i th component of $\mathbf{z}(\hat{\mathbf{x}}, \Delta t)$ is given by

$$z_i(\hat{\mathbf{x}}, \Delta t) = \sum_{k=1}^{p_i} \frac{\Delta t^k}{k!} L_f^k(c_i) \quad (18)$$

where $L_f^k(c_i)$ is the k th Lie derivative, defined by

$$L_f^0(c_i) = c_i, \quad L_f^k(c_i) = \frac{\partial L_f^{k-1}(c_i)}{\partial \hat{\mathbf{x}}} \mathbf{f}, \quad \text{for } k \geq 1 \quad (19)$$

The i th row of $S(\hat{\mathbf{x}})$ is given by

$$s_i = \left\{ L_{g_1}[L_f^{p_i-1}(c_i)], \dots, L_{g_l}[L_f^{p_i-1}(c_i)] \right\}, \quad i = 1, 2, \dots, m \quad (20)$$

where g_j is the j th column of $G(t)$, and the Lie derivative is defined by

$$L_{g_j}[L_f^{p_i-1}(c_i)] \equiv \frac{\partial L_f^{p_i-1}(c_i)}{\partial \hat{\mathbf{x}}} g_j, \quad j = 1, 2, \dots, l \quad (21)$$

Equation (20) is in essence a generalized sensitivity matrix for nonlinear systems. Therefore, given a state estimate at time t_k , then Eq. (16) is used to process the measurement at time t_{k+1} to find the $\mathbf{d}(t_k)$ to be used in $[t_k, t_{k+1}]$ to propagate the state estimate to time t_{k+1} . The weighting matrix W serves to weight the relative importance between the propagated model and measured quantities. If this matrix is set to zero, then no weight is placed on minimizing the model corrections so that a memoryless estimator is given.

ALLEGRO Algorithm

In the ALLEGRO algorithm, the attitude matrix is parameterized by the quaternion representation, defined as¹⁹

$$\mathbf{q} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \quad (22)$$

with

$$\mathbf{q}_{13} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \hat{\mathbf{e}} \sin(\theta/2) \quad (23a)$$

$$q_4 = \cos(\theta/2) \quad (23b)$$

where $\hat{\mathbf{e}}$ is a unit vector corresponding to the axis of rotation and θ is the angle of rotation. The quaternion satisfies a single constraint, given by

$$\mathbf{q}^T \mathbf{q} = \mathbf{q}_{13}^T \mathbf{q}_{13} + q_4^2 = 1 \quad (24)$$

The attitude matrix is related to the quaternion by

$$A(\mathbf{q}) = -\Xi^T(\mathbf{q})\Psi(\mathbf{q}) \quad (25)$$

with

$$\Xi(\mathbf{q}) \equiv \begin{bmatrix} q_4 I_{3 \times 3} + [\mathbf{q}_{13} \times] \\ \dots \dots \dots \\ -\mathbf{q}_{13}^T \end{bmatrix} \quad (26a)$$

$$\Psi(\mathbf{q}) \equiv \begin{bmatrix} -q_4 I_{3 \times 3} + [\mathbf{q}_{13} \times] \\ \dots \dots \dots \\ \mathbf{q}_{13}^T \end{bmatrix} \quad (26b)$$

From Eq. (3) it is clear that the quaternion representation leads to a loss function that is quartic in the quaternions. This is not equivalent to the familiar attitude determination loss function posed by Wahba.⁸ Therefore, in general, the GPS loss function poses a more difficult problem to solve than the standard vector-observation loss function in Wahba's problem.

In the ALLEGRO algorithm, it is assumed that the model is given by the quaternion kinematics model.¹⁹ This algorithm requires no dynamics model; it assumes that the attitude rate is adequately modeled by a constant model error \mathbf{d} between measurements, so that

$$\dot{\hat{\mathbf{q}}} = \frac{1}{2}\Xi(\hat{\mathbf{q}})\mathbf{d} \quad (27)$$

where $\hat{\mathbf{q}}$ denotes the determined quaternion. Because the phase difference measurements are used as the required tracking trajectories, the output vector in Eq. (12b) [dropping the subscript ij for the moment] is given by

$$\mathbf{c}(\hat{\mathbf{x}}) = \mathbf{b}^T A(\hat{\mathbf{q}})\mathbf{s} \quad (28)$$

The lowest-order time derivative of $\hat{\mathbf{q}}$ in Eq. (28) in which any component of \mathbf{d} first appears in Eq. (27) is one, so that $p_i = 1$. To derive the S matrix in Eq. (16), the following matrices are first defined:

$$\Omega(\mathbf{b}) \equiv \begin{bmatrix} -[\mathbf{b} \times] & \vdots & \mathbf{b} \\ \dots & \ddots & \dots \\ -\mathbf{b}^T & \vdots & 0 \end{bmatrix} \quad (29a)$$

$$\Gamma(\mathbf{s}) \equiv \begin{bmatrix} -[\mathbf{s} \times] & \vdots & -\mathbf{s} \\ \dots & \ddots & \dots \\ \mathbf{s}^T & \vdots & 0 \end{bmatrix} \quad (29b)$$

Substituting Eq. (25) into Eq. (28) and using $\Omega(\mathbf{b})\mathbf{q} = \Xi(\mathbf{q})\mathbf{b}$ and $\Gamma(\mathbf{s})\mathbf{q} = \Psi(\mathbf{q})\mathbf{s}$ leads to

$$\mathbf{c}(\hat{\mathbf{x}}) = \mathbf{q}^T \Omega(\mathbf{b})\Gamma(\mathbf{s})\mathbf{q} \quad (30)$$

The S matrix is formed by taking the partial of Eq. (30) with respect to \mathbf{q} and right multiplying the result by $\frac{1}{2}\Xi(\mathbf{q})$. Using $\Xi^T(\mathbf{q})\Gamma(\mathbf{s})\Xi(\mathbf{q}) = -[A(\mathbf{q})\mathbf{s} \times]$, expanding for all available baselines and sightlines, and using the sightlines at time t_{k+1} lead to

$$S = \begin{bmatrix} \mathbf{b}_1^T [A(\hat{\mathbf{q}}) \mathbf{s}_1^\Delta \times] \\ \vdots \\ \mathbf{b}_m^T [A(\hat{\mathbf{q}}) \mathbf{s}_n^\Delta \times] \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1^T A(\hat{\mathbf{q}}) [\mathbf{s}_1^\Delta \times] \\ \vdots \\ \mathbf{b}_m^T A(\hat{\mathbf{q}}) [\mathbf{s}_n^\Delta \times] \end{bmatrix} A^T(\hat{\mathbf{q}}) \quad (31)$$

where the superscript Δ denotes that the quantity is measured at time t_{k+1} (all other quantities are at time t_k). The right side of Eq. (31) will later be used in the covariance derivation. For a deterministic attitude solution, i.e., a memoryless approach, the weighting matrix W is set to zero in Eq. (16). The remaining quantities in Eq. (16) can be shown to be given by

$$\Lambda = \Delta t I_{3 \times 3} \quad (32a)$$

$$\hat{\mathbf{y}} = \begin{bmatrix} \mathbf{b}_1^T A(\hat{\mathbf{q}}) \mathbf{s}_1^\Delta \\ \vdots \\ \mathbf{b}_m^T A(\hat{\mathbf{q}}) \mathbf{s}_n^\Delta \end{bmatrix} \quad (32b)$$

$$\tilde{\mathbf{y}}(t_{k+1}) = [\Delta \tilde{\phi}_{11}^\Delta, \dots, \Delta \tilde{\phi}_{mn}^\Delta]^T \quad (32c)$$

$$R = \text{diag}[\sigma_{11}, \dots, \sigma_{mn}] \quad (32d)$$

$$\mathbf{z}(\hat{\mathbf{x}}_k, \Delta t) = \mathbf{0} \quad (32e)$$

Therefore, the following model error equation is developed:

$$\begin{aligned} \mathbf{d}(t_k) \equiv \mathbf{d}_k = & -\frac{1}{\Delta t} \left\{ \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} [A \mathbf{s}_j^\Delta \times] \mathbf{b}_i \mathbf{b}_i^T [A \mathbf{s}_j^\Delta \times]^T \right\}^{-1} \\ & \times \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} [A \mathbf{s}_j^\Delta \times] \mathbf{b}_i (\Delta \tilde{\phi}_{ij}^\Delta - \mathbf{b}_i^T A \mathbf{s}_j^\Delta) \end{aligned} \quad (33)$$

It should be noted that Eq. (33) represents an exact linearization for an interval Δt (Ref. 20). However, for practical applications the sampling interval should be well below Nyquist's limit.²¹ The determined quaternion can be found by integrating Eq. (27) from time t_k to t_{k+1} . Because \mathbf{d} is assumed constant over this interval, a discrete propagation for Eq. (27) can be used, given by

$$\hat{\mathbf{q}}_{k+1} = [\beta_k I_{4 \times 4} + \gamma_k \Omega(\rho_k)] \hat{\mathbf{q}}_k \quad (34)$$

where

$$\beta_k \equiv \cos\left(\frac{1}{2} \|\mathbf{d}_k\| \Delta t\right) \quad (35a)$$

$$\gamma_k \equiv \sin\left(\frac{1}{2} \|\mathbf{d}_k\| \Delta t\right) \quad (35b)$$

$$\rho_k = \mathbf{d}_k / \|\mathbf{d}_k\| \quad (35c)$$

To derive an attitude error covariance from Eq. (27), a propagated expression can be derived using a similar approach found in Ref. 22. The attitude error equation is given by

$$\delta \dot{\alpha} = -[\mathbf{d} \times] \delta \alpha + \delta \mathbf{d} \quad (36)$$

where $\delta \mathbf{d}$ is a model error perturbation (also, for the quaternion $2\delta \mathbf{q}_{13} \approx \delta \alpha$). The discrete propagation is given by

$$\delta \alpha_{k+1} = \Phi_k \delta \alpha_k + \Gamma_k \delta \mathbf{d}_k \quad (37)$$

where

$$\Phi_k = e^{-[\mathbf{d} \times] \Delta t} \quad (38a)$$

$$\Gamma_k = \int_0^{\Delta t} e^{-[\mathbf{d} \times] t} dt \quad (38b)$$

Next, the true output is given by using a first-order expansion in the predictive filter output,¹⁵ so that

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \Delta t \bar{S}_k \bar{\mathbf{d}}_k + \mathbf{v}_{k+1} \quad (39)$$

where \bar{S} and $\bar{\mathbf{d}}$ correspond to true quantities of S from Eq. (31) and \mathbf{d} corresponds to true quantities of S from Eq. (33), respectively. Therefore, the model error is given by

$$\mathbf{d}_k = (1/\Delta t) K_k (\mathbf{y}_k - \hat{\mathbf{y}}_k + \mathbf{v}_{k+1} + \Delta t \bar{S}_k \bar{\mathbf{d}}_k) \quad (40)$$

where

$$K_k \equiv (S_k^T R^{-1} S_k)^{-1} S_k^T R^{-1} \quad (41)$$

Next, using a small-angle perturbation in the attitude matrix, similar to Eq. (9), leads to

$$\mathbf{y}_k - \hat{\mathbf{y}}_k \approx S_k \delta \alpha_k \quad (42)$$

Now if $\delta \alpha_k$ is small, using the right-hand side of Eq. (31), the following approximation can be used:

$$\bar{S}_k \approx S_k (I_{3 \times 3} + [\delta \alpha_k \times]) \quad (43)$$

Therefore, because $K_k S_k = I_{3 \times 3}$, the model error equation is now given by

$$\mathbf{d}_k = \frac{\delta \alpha_k}{\Delta t} + \frac{K_k \mathbf{v}_{k+1}}{\Delta t} + (I_{3 \times 3} + [\delta \alpha_k \times]) \bar{\mathbf{d}}_k \quad (44)$$

Using $\delta \mathbf{d}_k = \bar{\mathbf{d}}_k - \mathbf{d}_k$ leads to the following error angle equation:

$$\delta \alpha_{k+1} = \Phi_k \delta \alpha_k - \frac{\Gamma_k \delta \alpha_k}{\Delta t} - \frac{\Gamma_k K_k \mathbf{v}_{k+1}}{\Delta t} + \Gamma_k [\bar{\mathbf{d}}_k \times] \delta \alpha_k \quad (45)$$

If Δt is small, as assumed in this approach, i.e., the sampling interval is well within Nyquist's limit, $\|\mathbf{d}_k\| \Delta t < \pi/10$ (Ref. 21), then the quantities in Eq. (38) can be approximated adequately by

$$\Phi_k \approx (I_{3 \times 3} - \Delta t [\mathbf{d}_k \times]) \quad (46a)$$

$$\Gamma_k \approx \Delta t I_{3 \times 3} \quad (46b)$$

Substituting these quantities into Eq. (45) leads to

$$\delta \alpha_{k+1} \approx -K_k \mathbf{v}_{k+1} \quad (47)$$

The cancellation of the terms in $\delta \alpha_k$ reflects that setting $W = 0$ in Eq. (16) gives a memoryless estimator. Now the attitude error covariance is given by

$$P_{k+1} \equiv E\{\delta \alpha_{k+1} \delta \alpha_{k+1}^T\} = K_k R K_k^T \quad (48)$$

Therefore, from the definitions of S_k , K_k , and R , the attitude error covariance expression for the ALLEGRO algorithm becomes

$$P_{k+1} = \left\{ \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} [A(\hat{\mathbf{q}}_k) \mathbf{s}_j^\Delta \times] \mathbf{b}_i \mathbf{b}_i^T [A(\hat{\mathbf{q}}_k) \mathbf{s}_j^\Delta \times]^T \right\}^{-1} \quad (49)$$

Note that the attitude matrix in Eq. (49) is evaluated at time t_k and that the sightlines are given at time t_{k+1} . This may be simplified by using the following attitude propagation, which is valid for small Δt :

$$A(\hat{\mathbf{q}}_{k+1}) = (I_{3 \times 3} - \Delta t [\mathbf{d}_k \times]) A(\hat{\mathbf{q}}_k) \quad (50)$$

The inverse recursion for $A(\hat{\mathbf{q}}_k)$ can be adequately approximated by

$$A(\hat{\mathbf{q}}_k) \approx (I_{3 \times 3} + \Delta t [\mathbf{d}_k \times]) A(\hat{\mathbf{q}}_{k+1}) \quad (51)$$

Substituting Eq. (51) into Eq. (49) leads to

$$P_{k+1} \approx \left[\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} (\xi_{ij}^\Delta) (\xi_{ij}^\Delta)^T \right]^{-1} \quad (52)$$

where

$$\xi_{ij}^\Delta \equiv [\mathbf{b}_i \times] (I_{3 \times 3} + \Delta t [\mathbf{d}_k \times]) A_{k+1} \mathbf{s}_j^\Delta \quad (53)$$

The term in Eq. (52) that involves $[\mathbf{d}_k \times]$ is typically three orders of magnitude less than the term that does not involve $[\mathbf{d}_k \times]$, and the

term that is quadratic in $[d_k \times]$ is typically six orders of magnitude less than the term that does not involve $[d_k \times]$. Thus, Eq. (52) reduces down to

$$P_{k+1} \approx \left\{ \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} [A(\hat{q}_{k+1}) s_j^\Delta \times] b_i b_i^T [A(\hat{q}_{k+1}) s_j^\Delta \times]^T \right\}^{-1} \quad (54)$$

Therefore, the attitude error covariance at time t_k is given by

$$P \approx \left\{ \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} [A s_j \times] b_i b_i^T [A s_j \times]^T \right\}^{-1} \quad (55)$$

This expression is equivalent to the optimal covariance shown by Eq. (8). Therefore, the ALLEGRO algorithm is in essence equivalent to solving the generalized loss function in Eq. (3). Although the approximation in Eq. (55) is valid only for small Δt , this poses no problem for typical onboard applications, e.g., for a typical vehicle in low Earth orbit undergoing motion of one revolution-per-orbit, a sampling interval of 100 s is more than sufficient for Eq. (55) to be a valid approximation. Also, the inverse in Eq. (33) is sufficient to determine P_{k+1} , as shown by Eqs. (49–55). Therefore, the ALLEGRO algorithm inherently computes the attitude error covariance as part of its solution. Finally, Ref. 23 shows an analysis of robustness with respect to initial condition errors. It is shown that the estimated error in the predictive filter is always bounded for any initial condition, which makes the ALLEGRO more robust than a linearized least-squares algorithm.

There are many advantages of the ALLEGRO algorithm over previous methods. These advantages include the following:

- 1) The ALLEGRO algorithm can provide estimates even when the baselines are coplanar, which is an advantage over the methods shown in Refs. 11 and 14 that convert the GPS problem into a form equivalent to Wahba's problem. Also, it has been shown in Ref. 11 that the attitude of a vehicle can be determined with a minimum of two baselines and two sightlines (to within a sign change). This is also true for the ALLEGRO algorithm, for which the solution will converge to the true attitude as long as the initial condition is in the correct hemisphere.
- 2) Unlike gradient based-methods the ALLEGRO algorithm is noniterative, which provides a more numerically stable algorithm.
- 3) The ALLEGRO algorithm is robust with respect to initial condition errors, which is an advantage over the linearized least-squares algorithm.
- 4) The computational burden of the ALLEGRO algorithm is low because the algorithm is easily programmable using Eqs. (33) and (34).

Hardware Simulation

A hardware simulation of a typical spacecraft attitude determination application was undertaken to demonstrate the performance of the ALLEGRO algorithm. For this simulation, a Northern Telecom 40 channel, 4-rf output STR 2760 unit was used to generate the GPS signals that would be received at a user specified location and velocity. The signals are then provided directly, i.e., they are not actually radiated, to a GPS receiver that has been equipped with software tracking algorithms that allow it to operate in space (see Fig. 2).

The receiver that was used was a Trimble TANS Vector; which is a 6-channel, 4-rf input multiplexing receiver that performs 3-axis attitude determination using GPS carrier phase and line-of-sight measurements. This receiver was modified in software at Stanford University and NASA Goddard Space Flight Center (GSFC) to allow it to operate in space. This receiver model has flown and operated successfully on several spacecraft, including: REX-II, OAST-Flyer, GANE, Orbcomm, Microlab, and others.

The simulated motion profile was for an actual spacecraft, the Small Satellite Technology Initiative (SSTI) Lewis satellite, which was launched on Aug. 22, 1997 (see Fig. 3). The orbit parameters used for the simulation are given in Table 1. This mission actually carried a GPS attitude determination experiment to assess the performance of the GPS attitude measurements on-orbit. Although the

Table 1 SSTI NASA Lewis Research Center orbit parameters

Parameter	Value
Semimajor axis a	6901.137 km
Inclination i	97.45 deg
Right ascension of ascending node (RAAN)	−157.1 deg
Eccentricity e	0.0001
Pointing profile	Earth pointed
Launch date	Aug. 22, 1997

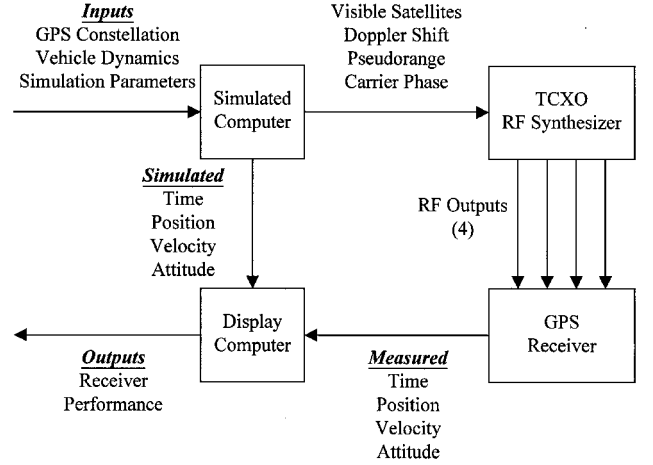


Fig. 2 Hardware simulation block diagram.



Fig. 3 SSTI Lewis on-orbit configuration (artist depiction).

spacecraft was lost due to a malfunction not related to the GPS experiment shortly after launch, this motion profile is nonetheless very representative of the types of attitude determination applications that are found on satellites.

The antenna separation distances are 0.61, 1.12, and 1.07 m, respectively. One antenna (in baseline 3) is located 0.23 m out of plane (below) the other three antennas. On the spacecraft, the antennas are mounted on pedestals with ground planes to minimize signal reflections and multipath. For the simulation, the signal was provided to the GPS receiver without multipath noise. The simulated SSTI Lewis spacecraft has four GPS antennas that form three baselines. The baseline vector components in wavelengths are given by

$$b_1 = \begin{bmatrix} 2.75 \\ 1.64 \\ -0.12 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0.00 \\ 6.28 \\ -0.17 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -3.93 \\ 3.93 \\ -1.23 \end{bmatrix} \quad (56)$$

Quantities such as line biases and integer ambiguities are first determined before the attitude determination algorithms are tested. The GPS raw measurements are processed at 1 Hz over a 40-min simulation. A plot of the number of available GPS spacecraft for the simulated run is shown in Fig. 4. During the beginning of the run there are 5–6 available spacecraft. At the end of the simulation this

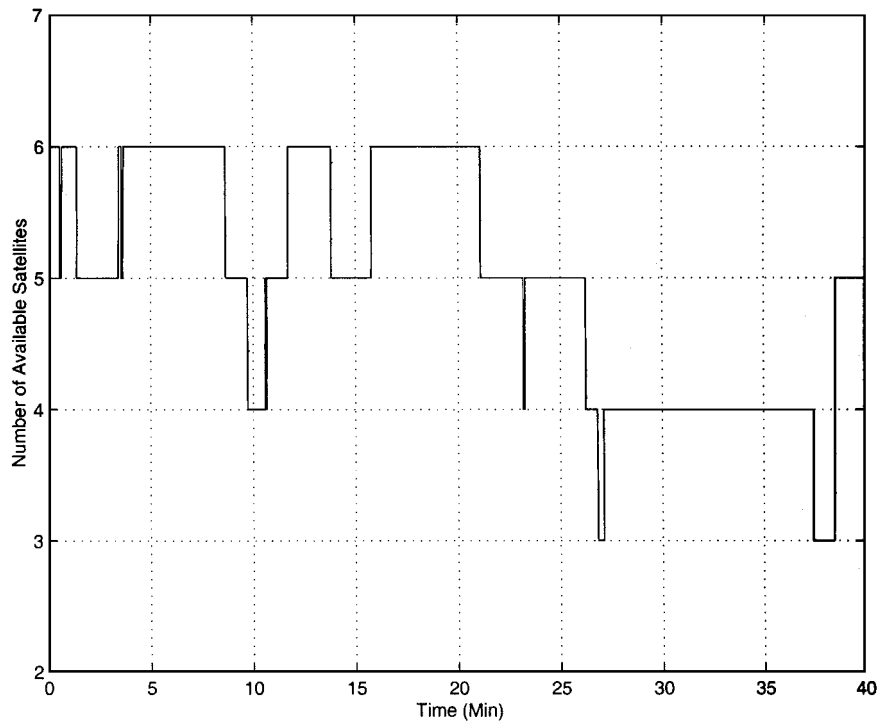


Fig. 4 Number of available GPS spacecraft.

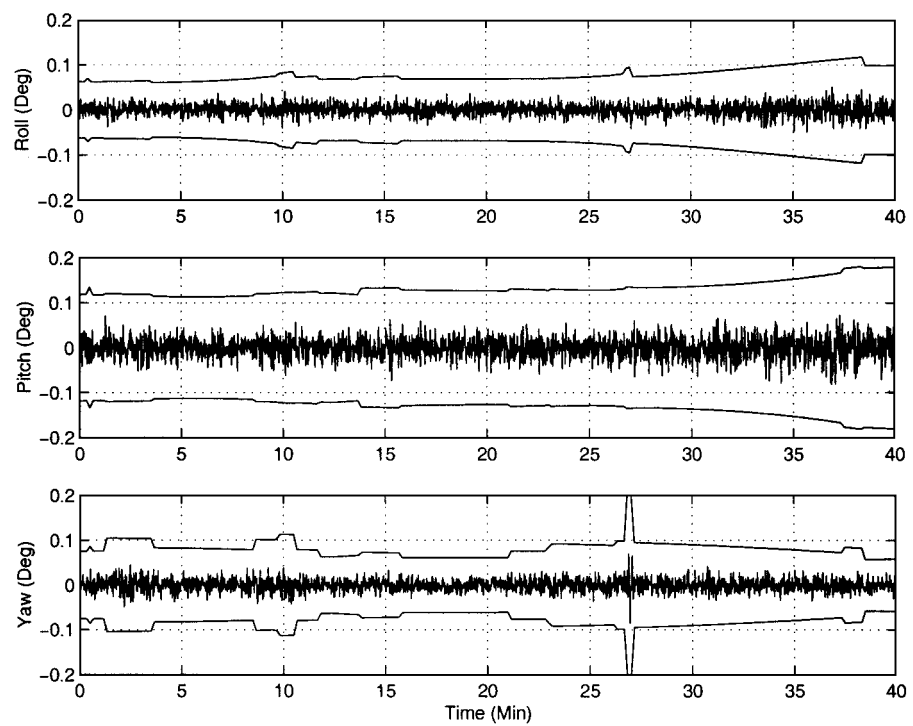


Fig. 5 Attitude errors and 3 sigma bounds for ALLEGRO.

drops down to about 4, which means that a degraded performance is possible (this also depends on the geometry of the spacecraft, see Ref. 13 for geometric dilution of precision). For the first simulation, the sightlines and baselines are used to form simulated phase difference measurements with Gaussian measurement errors. This is not a totally realistic simulation; however, it is useful to quantify the effectiveness of the ALLEGRO algorithm. A plot of the (roll, pitch, yaw) attitude errors with 3-sigma outliers using Eq. (55) is shown in Fig. 5. Clearly, the ALLEGRO algorithm provides estimates that agree with the optimal standard deviation predictions.

The remaining runs use the actual phase measurements from the receiver. This provides a more realistic scenario. The linearized least-squares approach using Eqs. (9–11) is also used to determine the attitude. Plots of the determined attitude using both algorithms are shown in Fig. 6; the results of the two algorithms are indistinguishable in this figure. The glitch between 10 and 15 min is due to receiver outages. To test the robustness of both algorithms, they were retested with a poor initial attitude guess. It should be noted that large initial attitude errors are not inconsistent with resolved integer phase ambiguities (see Ref. 7). A plot of the attitude errors during the iteration stage of the least-squares algorithm is shown

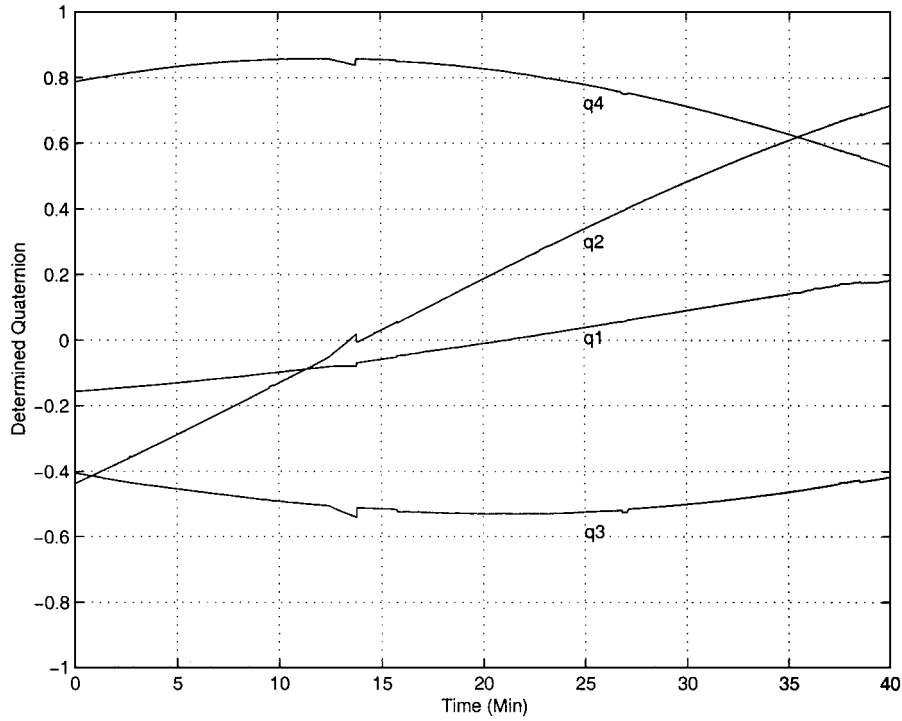


Fig. 6 Determined attitude using both ALLEGRO and linearized least-squares algorithms.

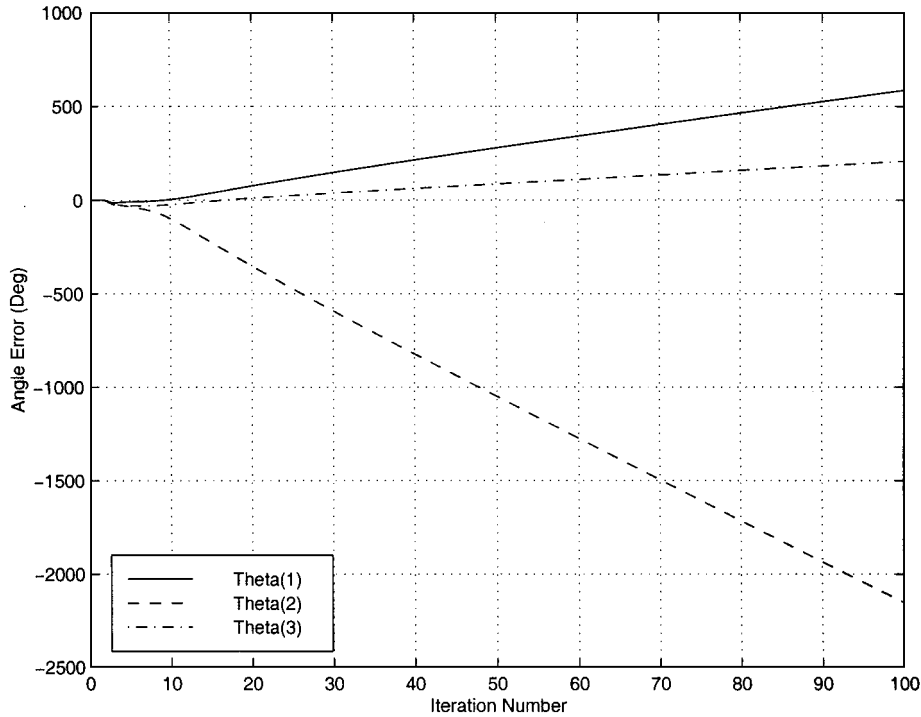


Fig. 7 Least-squares response to poor initial condition.

in Fig. 7. Clearly, the least-squares algorithm does not converge to the correct solution. This is due to the small-angle approximation in Eq. (9). The same initial condition is applied to the ALLEGRO algorithm. Because the ALLEGRO is sequential and noniterative, convergence is given over sampled intervals. A plot of the attitude errors is shown in Fig. 8. Clearly, the ALLEGRO algorithm converges to the correct solution (after three sampling intervals for this initial condition). To further test the robustness of the ALLEGRO algorithm, a Monte Carlo analysis has been performed using 1000 normalized random initial conditions. A plot of the convergence rates is shown in Fig. 9. In all cases, convergence is achieved within 19 sampling intervals (most converge within about 10 sampling intervals).

Finally, a test has been performed on the computational efficiency of the ALLEGRO algorithm. The number of floating-point operations (FLOPS) has been evaluated using MATLABTM. Both methods calculate the attitude error covariance as part of their solutions. A comparison with the least-squares algorithm is slightly misleading because ALLEGRO is noniterative. It has been determined that the only major difference between them is the ALLEGRO algorithm propagates a quaternion model. However, the computational expense of this propagation is smaller than 75 FLOPS, which is almost an order of magnitude less than doing a second iteration in the least-squares algorithm (even for only the two baseline and two sightline case). Therefore, the ALLEGRO algorithm is computationally comparable or better than the least-squares algorithm.

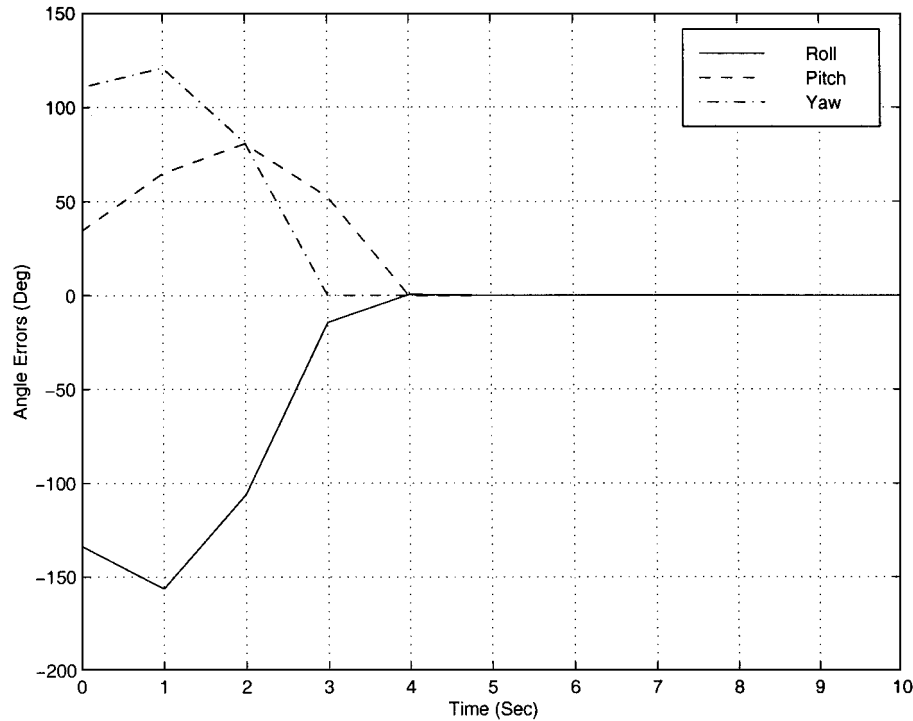


Fig. 8 ALLEGRO response to poor initial condition.

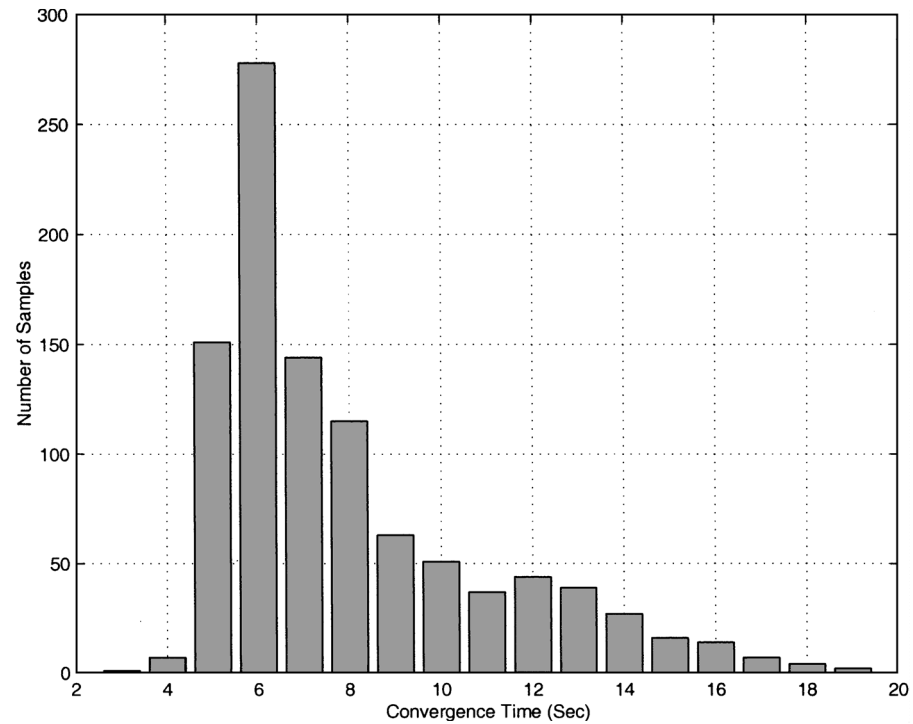


Fig. 9 Monte Carlo initial condition analysis using ALLEGRO.

Conclusions

A new optimal and efficient algorithm has been developed for attitude determination using GPS signals. It has been shown that the standard GPS loss function is inherently difficult to solve. The new noniterative algorithm provides sequential estimates using a recursive one-time step ahead approach. Attitude determination is accomplished by determining the angular velocity components used to propagate a simple quaternion kinematics model. An attitude error covariance expression has been derived for the new algorithm. This covariance has been shown to be equivalent to the optimal covariance, derived from maximum likelihood, if the sample interval is small enough (which poses no problem for most applications).

The algorithm was tested on a hardware simulator using an actual receiver. Results indicated that the new algorithm is computationally comparable to a linearized least-squares approach, while providing robustness with respect to initial conditions error. Therefore, the algorithm is exceptionally suitable for onboard applications.

Appendix: Alternative Covariance Derivation

In this section another approach for the attitude error covariance in the ALLEGRO algorithm is derived. Linearizing Eq. (34) in d_k gives

$$\hat{q}_{k+1} = [I_{4 \times 4} + (\Delta t/2)\Omega(d_k)]\hat{q}_k = \hat{q}_k + (\Delta t/2)\Xi(\hat{q}_k)d_k \quad (A1)$$

where the identity $\Omega(\mathbf{d}_k)\hat{\mathbf{q}}_k = \Xi(\hat{\mathbf{q}}_k)\mathbf{d}_k$ was used. Next, use $A = (I_{3 \times 3} - [\delta\alpha \times])A_{\text{true}}$, so that

$$\begin{aligned} \mathbf{b}_i^T A \mathbf{s}_j^\Delta &= \mathbf{b}_i^T A_{\text{true}} \mathbf{s}_j^\Delta - \mathbf{b}_i^T [\delta\alpha \times] A_{\text{true}} \mathbf{s}_j^\Delta \\ &= \Delta \bar{\phi}_{ij} - \delta\alpha^T \boldsymbol{\eta}_{ij} \end{aligned} \quad (\text{A2})$$

where $\boldsymbol{\eta}_{ij} = [A_{\text{true}} \mathbf{s}_j^\Delta \times] \mathbf{b}_i$ and $\Delta \bar{\phi}_{ij}$ denotes the true phase difference. Using $\Delta \bar{\phi}_{ij} - \Delta \bar{\phi}_{ij} = v_{ij}$ and substituting Eqs. (A2) and (33) into Eq. (A1) yield

$$\begin{aligned} \hat{\mathbf{q}}_{k+1} &= \left[I_{4 \times 4} - \frac{1}{2} \Omega(\delta\alpha_k) \right] \hat{\mathbf{q}}_k \\ &\quad - \frac{1}{2} \Xi(\hat{\mathbf{q}}_k) \left[\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} \boldsymbol{\eta}_{ij} \boldsymbol{\eta}_{ij}^T \right]^{-1} \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} \boldsymbol{\eta}_{ij} v_{ij} \end{aligned} \quad (\text{A3})$$

Now, using the true quaternion ($\bar{\mathbf{q}}$) representation

$$\bar{\mathbf{q}} = \left[I_{4 \times 4} - \frac{1}{2} \Omega(\delta\alpha) \right] \hat{\mathbf{q}} \quad (\text{A4})$$

yields the following approximation to within first order:

$$\hat{\mathbf{q}}_{k+1} = \bar{\mathbf{q}}_k - \frac{1}{2} \Xi(\bar{\mathbf{q}}_k) \left[\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} \boldsymbol{\eta}_{ij} \boldsymbol{\eta}_{ij}^T \right]^{-1} \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} \boldsymbol{\eta}_{ij} v_{ij} \quad (\text{A5})$$

Defining $\delta\mathbf{q} = \hat{\mathbf{q}} - \bar{\mathbf{q}}$ gives the following quaternion error covariance:

$$\mathbf{P}_{k+1}^{qq} = \frac{1}{4} \Xi(\bar{\mathbf{q}}_k) \left[\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} \boldsymbol{\eta}_{ij} \boldsymbol{\eta}_{ij}^T \right]^{-1} \Xi^T(\bar{\mathbf{q}}_k) \quad (\text{A6})$$

Therefore, using the same principles for the attitude error covariance derivation in Ref. 22 and from the analogy in Eqs. (49–54) gives

$$\mathbf{P} \approx \left[\sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}^{-2} [A \mathbf{s}_j \times] \mathbf{b}_i \mathbf{b}_i^T [A \mathbf{s}_j \times]^T \right]^{-1} \quad (\text{A7})$$

which is again the same expression as in Eqs. (8) and (55).

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